Complexity of the Game Domination Problem

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Abstract

The game domination number is a graph invariant that arises from a game, which is related to graph domination in a similar way as the game chromatic number is related to graph coloring. In this paper we show that verifying whether the game domination number of a graph is bounded by a given integer is PSPACE-complete. This contrasts the situation of the game coloring problem whose complexity is still unknown.

Key words: Domination game; Computational complexity; PSPACE-complete problems; POS-CNF problem;

AMS Subj. Class: 05C57, 68Q15, 05C69

1 Introduction

The domination game, introduced in [6], is played by two players on an arbitrary graph G. The two players are called Dominator and Staller, indicating the roles they are supposed to play in the game. They are taking turns choosing a vertex from G such that whenever they choose a vertex, the set of vertices dominated so far increases. The game ends when all vertices of G are dominated. The aim of Dominator is that the total number of moves played in the game is as small as possible, while Staller wishes to maximize this number. By *Game 1* we mean a game in which Dominator has the first move, while *Game 2* refers to a game in which Staller begins. Assuming that both players play optimally, the *game domination number* $\gamma_g(G)$ of a graph G denotes the number of moves played in Game 1, and the *Staller-start game domination number* $\gamma_g'(G)$ the number of moves played in Game 2.

Giving a formula for the exact value of $\gamma_g(G)$ and $\gamma_g'(G)$ is usually a difficult problem, and is resolved only for some very simple families of graphs G, such as paths and cycles [17], combs [18], and line graphs of complete multipartite graphs [20]. It was conjectured in [16] that the upper bound $\gamma_g(G) \leq 3/5|V(G)|$ holds for any isolate-free forest as well as for any isolate-free graph G. Whether this bound is true remains an open question although quite some progress was made. In the seminal paper [16], the conjecture was verified for forests in which each component is a caterpillar. Bujtás [9] confirmed the conjecture for the class of forests in which no two leaves are at distance 4 apart, while in [10] she proved upper bounds for $\gamma_g(G)$ which are better than $3/5|V(G)|$ as soon as the minimum degree of G is at least 3. We also add that in [8], large families of trees were constructed that attain the conjectured 3/5-bound and all extremal trees on up to 20 vertices were found.

The game domination number has been studied also from several additional aspects (see [5, 7, 11]), nevertheless the algorithmic complexity of determining $\gamma_g(G)$ for a given graph G was not yet studied. This comes with no surprise, considering that the complexity of the much older coloring game [2] has not been determined yet. Bodlaender [3] proved that a version of the coloring game where the order of the vertices to be colored is prescribed in advance is PSPACE-complete (see also [4]), but to the best of our knowledge no result is known for the standard coloring game.

In this paper we prove that the complexity of verifying whether the game domination number of a graph is bounded by a given integer is in the class of PSPACE-complete problems, implying that every problem solvable in polynomial space (possibly with exponential time) can be reduced to this problem. In particular, this shows that the game domination number of a graph is harder to compute than any other classical domination parameter (which are generally NP-hard), unless NP=PSPACE. The reduction we use can be computed with a working space of logarithmic size with respect to the entry, making this problem log-complete in PSPACE. (For additional problems that were recently proved to be PSPACE-complete see [1, 12, 13, 15].)

Throughout the paper we use the convention that $d_1, d_2, \ldots$ denotes the sequence of vertices chosen by Dominator and $s_1, s_2, \ldots$ the sequence chosen by Staller. Similarly, in the Staller-start game we use the notation $s'_1, s'_2, \ldots$ for the sequence chosen by Staller and $d'_1, d'_2, \ldots,$ for Dominator. A *partially-dominated graph* is a graph together with a declaration that some vertices are already dominated, that is, they need not be dominated in the rest of the game. For a vertex subset S of a graph G, let $G|S$ denote the partially dominated graph in which vertices from S are already dominated (note that S can be an arbitrary subset of $V(G)$, and not only a union of closed neighborhoods of some vertices).

In the following section, we present a reduction from the classical PSPACE-complete problem POS-CNF to a game domination problem where some vertices are set to be dominated before the game begins. Then, in Section 3, we describe how to extend the reduction to the Staller-start domination game and to the game on a graph not partially-dominated. We conclude with some open questions.

2 PSPACE complexity of the game domination problem

The game domination problem is the following:

GAME DOMINATION PROBLEM	
Input:	A graph G, and an integer m.
Question:	Is $\gamma_g(G) \leq m$?

To prove the complexity of the GAME DOMINATION PROBLEM, we propose a reduction from the POS-CNF problem, which is known to be log-complete in PSPACE [19]. In POS-CNF we are given a set of k variables, and a formula that is a conjunction of n disjunctive clauses, in which only positive variables appear (that is, no negations of variables). Two players alternate turns, Player 1 setting a previously unset variable TRUE, and Player 2 setting one FALSE. After all k variables are set, Player 1 wins if the formula is TRUE, otherwise Player 2 wins. In the proof of our main result, we transform a given formula $\mathcal{F}$ using k variables and n disjunctive clauses into a partially dominated graph $G_{\mathcal{F}}$, having $9k + n + 4$ vertices. We then prove that Player 1 has a winning strategy for a formula $\mathcal{F}$ if and only if $\gamma_g(G_{\mathcal{F}}) \leq 3k + 2$.

In the construction of $G_{\mathcal{F}}$, we use k copies of the widget graph W in correspondence with the k variables. The graph W is constructed from the disjoint union of the cocktail-party graph $K_6 - M$ on the vertex set $\{a_2, x, x', y, y', z\}$ with $a_2z, xx', yy' \notin E(W)$, and of the path $P : b_1a_1b_2$, by the addition of the edges b_1x, b_1x', b_2y, b_2y' and a_1z. Moreover, the vertices a_1 and a_2 are assumed to be dominated, that is, we are considering the partially dominated graph $W|\{a_1, a_2\}$. In Fig. 1 the graph W is shown, where the vertices a_1 and a_2 are filled black to indicate that they are assumed to be dominated.

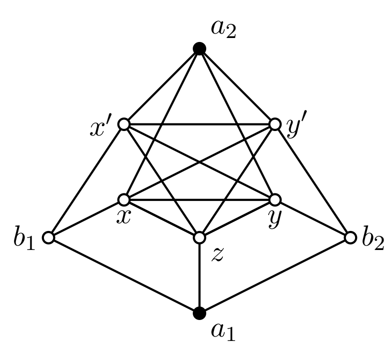

Figure 1: The widget graph W used to represent each variable

In the proof of the main result, we use several times the following properties of the graph W.

Observation 1 *Let W be the widget graph. Then* *(i)* $\gamma_g(W|\{a_1, a_2\}) = 3$ *and* a_1 *is an optimal first move;*

(ii) if $d_1 = a_1$*, and Staller passes her first move, then Dominator can finish the game in* $W|\{a_1, a_2\}$ *in two moves (by playing* a_2*);*

(iii) if $d_1 = a_1$*,* $s_1 = b_1$*, and Dominator passes his second move, then Staller can ensure four moves will be played in* $W|\{a_1, a_2\}$ *(e.g. by playing* y*);*

(iv) $\gamma_g'(W|\{a_1, a_2\}) = 3$ *and* b_1 *is an optimal first move for Staller;*

(v) if $s_1' = b_1$ *and Dominator responds playing* a_1 *or* a_2*, then Staller can enforce four moves are played in* $W|\{a_1, a_2\}$ *(by playing respectively* y *or* x*);*

(vi) if Staller starts and Dominator passes his first move, then after any second move of Staller, Dominator can finish the game in $W|\{a_1, a_2\}$ *with the third move, ensuring in addition that* a_1 *or* a_2 *is played.* □

Next we present a construction of the graph $G_{\mathcal{F}}$, when we are given a formula $\mathcal{F}$ with k variables and n clauses. We require that k is even, otherwise we add a variable that appears in no clause. For each variable X in $\mathcal{F}$ we take a copy W_X of the graph $W|\{a_1, a_2\}$ (that is, we assume that a_1 and a_2 in the copy W_X are dominated in $G_{\mathcal{F}}$). For each disjunctive clause $\mathcal{C}_i$ in the formula we add a vertex c_i, and for each X that appears in $\mathcal{C}_i$ we make c_i adjacent to both a_1 and a_2 from the copy of W_X. Next, we add edges c_ic_j between each two vertices, corresponding to disjunctive clauses $\mathcal{C}_i, \mathcal{C}_j$ that appear in $\mathcal{F}$. Hence the vertices c_i, $1 \le i \le n$, induce a clique Q of size n. Finally, we add a copy $P : p_1p_2p_3p_4$ of a path P_4, and add edges p_1c_i and p_4c_i for $1 \le i \le n$. See Fig. 2 for an example of the construction.

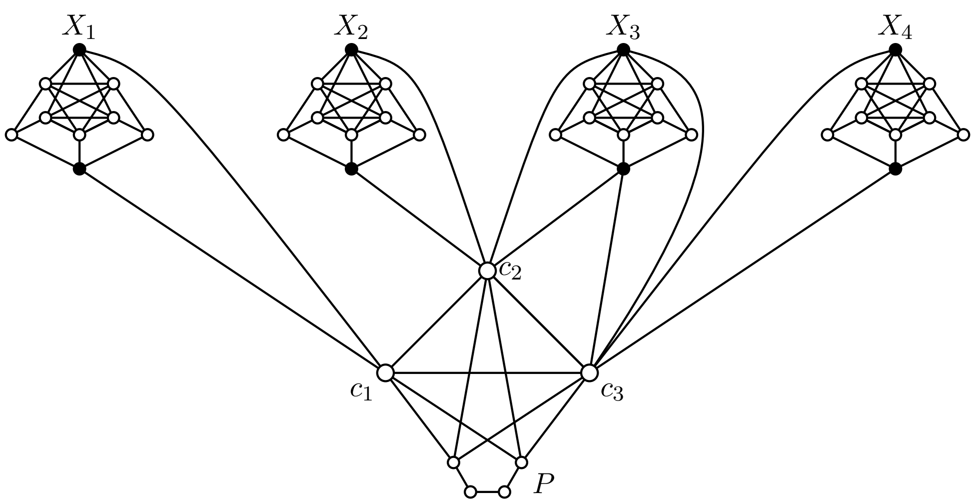

Figure 2: Example of the graph for formula $X_1 \wedge (X_2 \vee X_3) \wedge (X_3 \vee X_4)$

We call W_X a *widget subgraph* of $G_{\mathcal{F}}$, and in the notation for vertices in W_X we add X as an index to a vertex from W_X. For instance, a vertex that corresponds to a_1 in a widget subgraph W_X will be denoted by $a_{1,X}$, while the vertex that corresponds to z in this subgraph will be denoted by z_X.

The following observation will also be useful when the game is played on $G_{\mathcal{F}}$.

Observation 2 *Let H be a graph isomorphic to the subgraph of $G_{\mathcal{F}}$, induced by the vertices from $Q \cup P$, and let $S \subseteq V(Q)$ be some vertices already dominated (see Fig. 3).*

(i) If Q is not entirely dominated, that is, if $S \neq V(Q)$, then $\gamma_g(H|S) = 3$.

(ii) If $S = V(Q)$, then $\gamma_g(H|S) = 2$.

(iii) For any $S \subseteq V(Q)$, $\gamma_g'(H|S) = 2$.

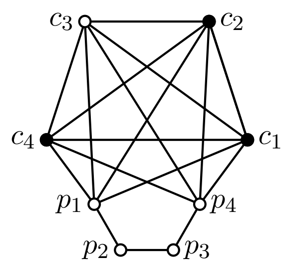

Figure 3: The graph $H|\{c_1, c_2, c_4\}$ for $n = 4$

Theorem 3 *Player 1 has a winning strategy for a formula $\mathcal{F}$ in the POS-CNF game if and only if $\gamma_g(G_{\mathcal{F}}) \leq 3k + 2$.*

Proof.

We first assume that Player 1 has a winning strategy for $\mathcal{F}$ in the POS-CNF game, and give a strategy of Dominator that ensures at most $3k + 2$ moves will be played in Game 1 on $G_{\mathcal{F}}$. To describe the strategy of Dominator in $G_{\mathcal{F}}$, we use a simultaneously played POS-CNF game on $\mathcal{F}$.

The first move of Dominator is to play $a_{1,X}$ where X is the first variable Player 1 would set TRUE in the POS-CNF game. Then whenever Staller makes a move which is the first move in a widget subgraph W_Y, Dominator considers this move as Player 2 setting Y FALSE in the POS-CNF game, and follows the POS-CNF winning strategy of Player 1 (playing $a_{1,X}$ where X is the next variable he would set TRUE in the POS-CNF game) as long as there are undefined variables. If Staller responds in a widget subgraph where one move was played already, Dominator answers in the same widget subgraph W_X to guarantee that no more than three moves are played in W_X (preventing situation from Observation 1(iii), if X is a variable set TRUE). If X is a variable set

FALSE, Dominator plays $a_{1,X}$ or $a_{2,X}$, which is possible by Observation 1(vi) (otherwise Staller could be allowed to play $a_{1,X}$ later on, if some adjacent vertices in Q were still undominated[1]). This way, he ensures that no more than three moves are played in each widget subgraph. If Staller played in $Q \cup P$, then using Observation 2(iii), Dominator finishes dominating $Q \cup P$ in the next move.

Suppose now that Staller did not play in $Q \cup P$, and that all variables are set TRUE or FALSE. Since Dominator followed POS-CNF strategy, $\mathcal{F}$ is true and all vertices in Q are thus dominated. Then Dominator can safely play p_2 and ensure no more than two moves are played in $Q \cup P$ (by Observation 2(ii)). Recalling that he also ensures at most three moves are played in each of the widget subgraphs (also in widget subgraphs that may not be entirely dominated yet), this implies $\gamma_g(G_{\mathcal{F}}) \leq 3k + 2$.

We now propose a strategy of Staller that ensures at least $3k + 3$ moves will be played in $G_{\mathcal{F}}$ when Player 2 has a winning strategy for $\mathcal{F}$ in the POS-CNF game.

The basic part of the strategy of Staller is that whenever Dominator makes a move in a widget subgraph W_X in which no move was made before, she responds in the same widget subgraph W_X by playing $b_{1,X}$ or $b_{2,X}$, ensuring three moves are played in W_X (by Observation 1(i)). Note that she has to respond or there is a possibility that only two moves will be played in the widget subgraph by Observation 1(ii).

To describe the rest of Staller's strategy in $G_{\mathcal{F}}$, we also use a simultaneously played POS-CNF game on $\mathcal{F}$. Whenever Dominator makes a move on $a_{1,X}$ or $a_{2,X}$ in some widget subgraph W_X, Staller considers Player 1 set the variable X TRUE in the POS-CNF game on $\mathcal{F}$. Then, when Staller's move is not forced by the basic part of her strategy, she will play in the widget subgraph W_Y that corresponds to the variable Y Player 2 would set FALSE in the POS-CNF game, as explained later.

Note that after Dominator plays $a_{1,X}$ or $a_{2,X}$ as the first move in some widget subgraph W_X, Staller has to respond in W_X by the basic part of her strategy, yielding the possibility for Dominator to set another variable TRUE (and thus Player 1 would be cheating in the POS-CNF game). However, if Dominator does not play in W_X, there is a threat that four moves will be played in W_X by Observation 1(iii). Suppose the next move of Dominator is not in W_X. As long as the next move of Dominator is in some W_Y in which no moves were made before, Staller must respond in W_Y (by the basic part of her strategy) to ensure three moves will be played in this W_Y. In the meantime, Dominator may have created some more threats by playing some $a_{1,Y}$ or $a_{2,Y}$. (Alternatively, his move could be a first move in a widget subgraph W_Z, different from $a_{i,Z}$. The response of Staller playing $b_{1,Z}$ or $b_{2,Z}$ in all cases implies that at least three moves will be played in W_Z.) Eventually, Dominator makes a move that does not force an immediate answer from Staller (that is, the third move in some widget subgraph W_Y or a move in $Q \cup P$). Unless he eliminates all threats with that move, Staller can use one threat (say in W_Y) and enforce four vertices are played in the widget subgraph W_Y. In any case, at least two moves must be played in $Q \cup P$ to dominate P, so at least $3k + 3$ moves will be played in the game, as desired.

[1]This is actually why such a complicated widget is needed for the variables, otherwise two adjacent vertices could serve as a widget subgraph.

If Dominator was able to eliminate all threats, it means that only one variable was set TRUE. Then Staller can select a variable Y according to the strategy of Player 2 in the POS-CNF game, and set it FALSE either by playing $b_{1,Y}$ as the first move in W_Y (forcing three moves by Observation 1(iv)), or by playing a third move in W_Y, such that none of these three moves is $a_{1,Y}$ or $a_{2,Y}$. Observe that if Dominator tries to re-set the variable TRUE by playing $a_{1,Y}$ or $a_{2,Y}$, then by Observation 1(v), Staller can immediately enforce four moves to be played in W_Y. If Dominator plays any other second move in W_Y, Staller just answers in W_Y, avoiding $a_{1,Y}$ and $a_{2,Y}$, and then all vertices of W_Y are dominated.

Eventually, Dominator plays in $Q \cup P$ (note that since k is even, Staller by her strategy forces Dominator to play first in $Q \cup P$). Then since Player 2 wins the POS-CNF game, not all vertices of Q are dominated at that point. Thus by Observation 2(i), Staller can ensure three moves are played in $Q \cup P$ and this makes a total of $3k + 3$ moves, completing the proof. □

3 Conclusions and open problems

First observe that the reduction from POS-CNF to Game Domination Problem can be computed with a working space of size $O(\log(k + n))$; giving an explicit algorithm is a routine work. Therefore, recalling that POS-CNF is log-complete in PSPACE [19], we get:

Corollary 4 Game Domination Problem *is log-complete in PSPACE.*

We can modify the above reduction to prove the PSPACE-completeness of the Staller-start game domination problem. Given a formula $\mathcal{F}$, consider the formula $\mathcal{F}'$, where we add a variable X_0 that we insert into every clause. Clearly, if Player 2 starts the POS-CNF game on $\mathcal{F}'$, he must set variable X_0 FALSE as his first move to have a chance to win. Then, the winner of the POS-CNF game on the formula $\mathcal{F}'$ where Player 2 starts, is the same as the winner of the POS-CNF game on $\mathcal{F}$. Using this knowledge, it is straightforward to use the above reduction for the Staller-start domination game.

Another natural question is whether having a partially dominated graph is necessary for the reduction. We describe now how to build a graph where no vertices are assumed already dominated. Consider a formula $\mathcal{F}$. First add a variable X_0 and modify $\mathcal{F}$ into a formula $\mathcal{F}'$ as described in the previous paragraph. Now we take the corresponding graph $G_{\mathcal{F}'}|S$ where S contains the vertices $a_{1,X}, a_{2,X}$ for all variables X (including X_0). Add to $G_{\mathcal{F}'}$ (where S is no longer dominated) a star $K_{1,3}$ with center v, and add an edge between v and each vertex in S; we denote the resulting graph $G'_{\mathcal{F}}$. We observe that $\gamma_g(G'_{\mathcal{F}}) \le 3k + 6$ if and only if Player 1 wins the POS-CNF game on $\mathcal{F}$ (Dominator must choose v for his first move, and Staller has to answer with b_{1,X_0}). Similarly, $\gamma'_g(G'_{\mathcal{F}}) \le 3k + 7$ if and only if Player 1 wins the POS-CNF game on $\mathcal{F}$.

A related problem is to try to find families of graphs where the game domination number can be computed efficiently. Such families obviously include those where the exact formula is known (as already mentioned, these include paths, cycles, combs, and line graphs of complete multipartite graphs). Another family where we expect the domination game number can be computed in polynomial time is the class of proper interval graphs. For these graphs, it looks like both players' strategy can be described by a greedy algorithm, though we did not manage to prove it. Hence we pose:

Question 1 *Can the game domination number of (proper) interval graphs be computed in polynomial time?*

In particular, interval graphs are also dually chordal graphs, which are proven in [11] to be the so-called no-minus graphs (that is, graphs G for which for all subsets of vertices S, $\gamma_g(G|S) \leq \gamma'_g(G|S)$). In that paper, stronger relations between the game domination number of the disjoint union of two graphs and the game domination number of the components are given for no-minus graphs, which could prove useful in Question 1. On the other hand, it seems likely that the decision problem remains PSPACE-complete even when restricted to split graphs, which are also proven to be no-minus in [11]. Note that this would be a similar dichotomy as proved in [14] for the Role Assignment problem which can be solved in polynomial time on proper interval graph and is Graph Isomorphism-hard on chordal graphs.

Observe that the domination game on split graphs transposes to a game on hypergraphs where Dominator chooses edges and Staller chooses vertices (not chosen before nor belonging to a chosen edge), and the game ends after all vertices either are chosen or belong to a chosen edge. The aim of Dominator is again to finish the game as soon as possible, while Staller tries to have the game last for as long as possible. The game where both players must choose hyperedges may be of independent interest.

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